A RATIO OF ALTERNANTS FORMULA FOR LOOP SCHUR FUNCTIONS

GABRIEL FRIEDEN

ABSTRACT. We prove a ratio of alternants formula for loop Schur functions, and we give a new proof of the loop Murnaghan-Nakayama rule.

1. Introduction

In [LP12], T. Lam and P. Pylyavskyy introduced a generalization of symmetric functions, which they called loop symmetric functions. These are polynomials in mn variables, which are invariant under a certain birational action of the symmetric group S_m. When $n = 1$, we recover the ordinary symmetric polynomials in m variables, and the birational action of S_m reduces to the ordinary permutation action on m variables.

Lam and Pylyavskyy defined loop analogues of the elementary and homogeneous symmetric functions, and they used "colored" Young tableaux to give a combinatorial definition of loop Schur functions. They proved a Jacobi-Trudi formula expressing the loop Schur functions as determinants of matrices consisting of loop homogeneous symmetric functions. Classically, Schur polynomials were first defined as a ratio of two alternants, and the Jacobi-Trudi and combinatorial interpretations came later, so it is natural to wonder if loop Schur functions can be expressed as a ratio of "loop alternants." Furthermore, the ratio of alternants formula is a useful tool for proving the Murnaghan-Nakayama rule and the Pieri rule. In [Lam12], a ratio of alternants formula for loop Schur functions was stated without proof, along with its corollary, the loop Murnaghan-Nakayama rule (the Pieri rule doesn't naturally generalize to the loop setting). In this paper we will prove the loop ratio of alternants formula and deduce from it the loop Murnaghan-Nakayama rule. (D. Ross has given a combinatorial proof of the loop Murnaghan-Nakayama rule in [Ros14].)

Lam and Pylyavskyy originally introduced loop symmetric functions in their study of total positivity in loop groups ([LP12]). The birational S_m action turns out to be the geometric R-matrix for certain affine geometric crystals, which has been studied previously by several authors ([Yam01], [BKb], [Eti03]). The tropicalization of this action gives a piecewise-linear formula for the (combinatorial) R-matrix for tensor products of single-row Kirillov-Reshetikhin crystals of type $A_n^{(1)}$ [HHI+01]. One important quantity associated to such tensor products is the intrinsic energy, which is invariant under the action of the R-matrix. It was shown in [LP13b] that the intrinsic energy is given by the tropicalization of a loop Schur function. Loop symmetric functions have also found application in the theory of the box-ball-system, a well-known discrete integrable system. Formulas for the scattering of a given state into solitons are given by tropicalizations of a variant of loop Schur functions, called cylindrical loop Schur functions ([LPS]). See [Lam12] for a survey of these and other applications of loop symmetric functions.

The organization of this paper is as follows. In Section 2, we define the ring of loop symmetric functions, and the various special classes of these functions (elementary, homogeneous,

power sum, Schur). We also define the birational S_m action and give its key properties. Section 3 contains statements and proofs of the alternants formula (Theorem 3.2) and the loop Murnaghan-Nakayama rule (Theorem 3.4).

I would like to thank my advisor, Thomas Lam, for introducing me to loop symmetric functions and the applications discussed above.

2. Background on Loop Symmetric Functions

2.1. **Whirls and Curls.** In [LP12], the authors define two types of infinite periodic matrices, called whirls and curls. We will review the definitions here.

Definition 2.1. Given a matrix $A = (a_{ij})$, define $A^c = (a'_{ij})$ by $a'_{ij} = (-1)^{i+j}a_{ij}$. If A is invertible, define $A^{-c} = (A^{-1})^c$.

Lemma 2.2. *For any $n \times n$ matrices A and B, we have $(AB)^c = A^cB^c$. If A is invertible, then $(A^c)^{-1} = (A^{-1})^c = A^{-c}$.*

Proof. We have $(A^cB^c)_{ij} = \sum(-1)^{i+k}A_{ik}(-1)^{k+j}B_{kj} = (-1)^{i+j}\sum A_{ik}B_{kj} = ((AB)^c)_{ij}$. The second statement follows from the first, since $(A^{-1})^cA^c = I^c = I$, where I is the identity matrix. □

Definition 2.3. Fix $n \geq 1$, and let $a_1, ..., a_n$ be indeterminates. The *whirl* $M(a_1, ..., a_n)$ is the infinite matrix $(M_{ij})_{1\leq i,j<\infty}$ with $M_{ii} = 1$, $M_{i,i+1} = a_i$ (interpret the subscripts on the $a's$ modulo n), and all other entries zero. The *curl* $N(a_1, ..., a_n)$ is defined by $N(a_1, ..., a_n) = M(a_1, ..., a_n)^{-c}$.

Example 2.4. When $n = 3$, we have

$$M(a_1,a_2,a_3) = \begin{pmatrix} 1 & a_1 & 0 & 0 & 0 & \\ 0 & 1 & a_2 & 0 & 0 & \\ 0 & 0 & 1 & a_3 & 0 & \cdots \\ 0 & 0 & 0 & 1 & a_1 & \\ 0 & 0 & 0 & 0 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

and

$$N(a_1,a_2,a_3) = \begin{pmatrix} 1 & a_1 & a_1a_2 & a_1a_2a_3 & a_1^2a_2a_3 & \\ 0 & 1 & a_2 & a_2a_3 & a_1a_2a_3 & \\ 0 & 0 & 1 & a_3 & a_1a_3 & \cdots \\ 0 & 0 & 0 & 1 & a_1 & \\ 0 & 0 & 0 & 0 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

Since whirls and curls are upper triangular, we may multiply them just like finite matrices, and Lemma 2.2 still holds.

2.2. **Loop Elementary and Homogeneous Symmetric Functions.** For the rest of this paper, fix integers $m \geq 1$ and $n \geq 1$. For $i = 1, ..., m$ and $j \in \mathbb{Z}/n\mathbb{Z}$, let $x_i^{(j)}$ be an indeterminate. We interpret the superscript modulo n, and we think of it as a "color." Let $\mathbb{Q}(x_i^{(j)})$ denote the field of rational functions in these mn variables. Let $M(\mathbf{x}_i) = M(x_i^{(1)}, \ldots, x_i^{(n)})$, and $N(\mathbf{x}_i) = N(x_i^{(1)}, \ldots, x_i^{(n)})$.

Definition 2.5. Set $A = M(\mathbf{x}_1)M(\mathbf{x}_2)\cdots M(\mathbf{x}_m)$, and $B = A^{-c} = N(\mathbf{x}_m)N(\mathbf{x}_{m-1})\cdots N(\mathbf{x}_1)$. Define the *loop elementary symmetric functions* $e_k^{(r)}(\mathbf{x}_1,\dots,\mathbf{x}_m)$ $(k \geq 0, r \in \mathbb{Z}/n\mathbb{Z})$ by

$$e_{j-i}^{(i)}(\mathbf{x}_1,\dots,\mathbf{x}_m) = A_{ij}$$

and the *loop homogeneous symmetric functions* $h_k^{(r)}(\mathbf{x}_1,\dots,\mathbf{x}_m)$ $(k \geq 0, r \in \mathbb{Z}/n\mathbb{Z})$ by

$$h_{j-i}^{(i)}(\mathbf{x}_1,\dots,\mathbf{x}_m) = B_{ij}.$$

Example 2.6. When $n = 2$ and $m = 3$, we have

$$M(\mathbf{x}_1)M(\mathbf{x}_2)M(\mathbf{x}_3) =$$
$$\begin{pmatrix} 1 & x_1^{(1)}+x_2^{(1)}+x_3^{(1)} & x_1^{(1)}x_2^{(2)}+x_2^{(1)}x_3^{(2)}+x_1^{(1)}x_3^{(2)} & x_1^{(1)}x_2^{(2)}x_3^{(1)} & \\ 0 & 1 & x_1^{(2)}+x_2^{(2)}+x_3^{(2)} & x_1^{(2)}x_2^{(1)}+x_2^{(2)}x_3^{(1)}+x_1^{(2)}x_3^{(1)} & \cdots \\ 0 & 0 & 1 & x_1^{(1)}+x_2^{(1)}+x_3^{(1)} & \\ 0 & 0 & 0 & 1 & \\ & & \vdots & & \ddots \end{pmatrix}$$

and

$$N(\mathbf{x}_3)N(\mathbf{x}_2)N(\mathbf{x}_1) =$$
$$\begin{pmatrix} 1 & x_1^{(1)}+x_2^{(1)}+x_3^{(1)} & x_1^{(1)}x_1^{(2)}+x_2^{(1)}x_2^{(2)}+x_3^{(1)}x_3^{(2)}+x_2^{(1)}x_1^{(2)}+x_3^{(1)}x_2^{(2)}+x_3^{(1)}x_1^{(2)} & \\ 0 & 1 & x_1^{(2)}+x_2^{(2)}+x_3^{(2)} & \cdots \\ 0 & 0 & 1 & \\ & & \vdots & \ddots \end{pmatrix}.$$

From the top rows of these matrices we can read off

$$e_1^{(1)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = h_1^{(1)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = x_1^{(1)}+x_2^{(1)}+x_3^{(1)}$$
$$e_2^{(1)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = x_1^{(1)}x_2^{(2)}+x_2^{(1)}x_3^{(2)}+x_1^{(1)}x_3^{(2)}$$
$$e_3^{(1)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = x_1^{(1)}x_2^{(2)}x_3^{(1)}$$
$$h_2^{(1)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = x_1^{(1)}x_1^{(2)}+x_2^{(1)}x_2^{(2)}+x_3^{(1)}x_3^{(2)}+x_2^{(1)}x_1^{(2)}+x_3^{(1)}x_2^{(2)}+x_3^{(1)}x_1^{(2)}$$
$$\text{etc.}$$

The following proposition gives similar formulas for all loop elementary and homogeneous symmetric functions, which reduce to the familiar formulas for ordinary elementary and homogeneous symmetric functions when colors are ignored (or, equivalently, when $n = 1$).

Proposition 2.7. *We have*

(a) $e_k^{(r)}(\mathbf{x}_1,\dots,\mathbf{x}_m) = \sum x_{i_1}^{(r)}x_{i_2}^{(r+1)}\cdots x_{i_k}^{(r+k-1)}$
where the sum is over strictly increasing sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq m$.

(b) $h_k^{(r)}(\mathbf{x}_1,\dots,\mathbf{x}_m) = \sum x_{i_k}^{(r)}x_{i_{k-1}}^{(r+1)}\cdots x_{i_1}^{(r+k-1)}$
where the sum is over weakly decreasing sequences $m \geq i_k \geq i_{k-1} \geq \cdots \geq i_1 \geq 1$.

The proof of this proposition is a straightforward induction on m, which is carried out in [LP12] (note, however, that the authors of that paper use the opposite ordering of the variables, so that their loop homogeneous symmetric functions are labeled by increasing sequences, and their "mirror" loop elementary symmetric functions are our loop elementary symmetric functions).

Remark 2.8. Observe the appearance of *decreasing* sequences in part (b) of Proposition 2.7. This is due to the fact that the homogenous symmetric functions are entries of the product $N(\mathbf{x}_m)\cdots N(\mathbf{x}_1)$. If increasing sequences were used instead, the color indexing of the loop homogeneous functions would have to be changed, and we feel that this convention is the least awkward choice.

The $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_i^{(j)})$ generated by the loop elementary symmetric functions is called the *ring of loop symmetric functions*, and is denoted LSym. Observe that if $n = 1$, then the loop elementary symmetric functions are simply the ordinary elementary symmetric polynomials $e_k(x_1, ..., x_m)$. Indeed, under the correspondence between power series and Toeplitz matrices, the matrix A in Definition 2.5 corresponds to the power series $E(t) = \prod_{i=1}^{m}(1 + x_i t)$, whose coefficients are the elementary symmetric polynomials.

Similarly, the loop homogeneous symmetric functions reduce to the ordinary homogeneous symmetric polynomials in m variables when $n = 1$. The matrix B in Definition 2.5 corresponds to the power series $H(t) = \prod_{i=1}^{m} \frac{1}{1-x_i t} = \frac{1}{E(-t)}$, whose coefficients are the homogenous symmetric polynomials. The loop homogeneous symmetric functions also generate LSym, although we will not prove this here.

The ordinary symmetric polynomials in m variables are invariant under an action of the symmetric group S_m, namely, the action that permutes the variables. The loop symmetric functions are invariant under a more complicated S_m action, as will be explained in the next section.

2.3. **The Birational S_m Action.** In [LP12], the authors define a birational map $R : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ with the property that if $(\mathbf{y}', \mathbf{x}') = R(\mathbf{x}, \mathbf{y})$, then $M(\mathbf{x})M(\mathbf{y}) = M(\mathbf{y}')M(\mathbf{x}')$. This map, called the birational R-matrix, plays a central role in their study of factorizations of matrices of power series into products of whirls and curls, as it allows whirls and curls to be "commuted" past one another. Since the loop elementary symmetric functions are defined to be the entries of a product of whirls, it follows that they are invariant under applying this R-matrix to each adjacent pair of variables $(\mathbf{x}_i, \mathbf{x}_{i+1})$. We will review the definition and key properties of the birational R-matrix below.

Definition 2.9. Let $\mathbf{x} = (x^{(1)}, \ldots, x^{(n)})$ and $\mathbf{y} = (y^{(1)}, \ldots, y^{(n)})$. For $r = 1, ..., n$, define

$$\kappa^{(r)}(\mathbf{x}, \mathbf{y}) = h_{n-1}^{(r+1)}(\mathbf{x}, \mathbf{y})$$
$$= x^{(r+1)}x^{(r+2)}\cdots x^{(r+n-1)} + y^{(r+1)}x^{(r+2)}\cdots x^{(r+n-1)} + \cdots + y^{(r+1)}y^{(r+2)}\cdots y^{(r+n-1)}$$

where the superscripts are interpreted modulo n.

Example 2.10. When $n = 4$, we have

$$\kappa^{(2)}(\mathbf{x}, \mathbf{y}) = x^{(3)}x^{(4)}x^{(1)} + y^{(3)}x^{(4)}x^{(1)} + y^{(3)}y^{(4)}x^{(1)} + y^{(3)}y^{(4)}y^{(1)}.$$

Definition 2.11. Let $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(n)})$ for $i = 1, ..., m$, and let $\mathbb{Q}(x_i^{(j)})$ denote the field of rational functions in the mn variables $x_i^{(j)}$, as in the previous section. For $i = 1, ..., m-1$ and $j = 1, ..., n$, define $\kappa_i^{(j)} = \kappa^{(j)}(\mathbf{x}_i, \mathbf{x}_{i+1})$ (as usual, the superscript is interpreted modulo n). Let $s_i : \mathbb{Q}(x_i^{(j)}) \to \mathbb{Q}(x_i^{(j)})$ be the unique $\mathbb{Q}$-algebra homomorphism such that

$$s_i(x_i^{(j)}) = x_{i+1}^{(j+1)} \frac{\kappa_i^{(j+1)}}{\kappa_i^{(j)}} \qquad \text{and} \qquad s_i(x_{i+1}^{(j)}) = x_i^{(j-1)} \frac{\kappa_i^{(j-1)}}{\kappa_i^{(j)}}$$

and s_i fixes $x_k^{(j)}$ for $k \neq i, i+1$. Given a matrix A with entries in $\mathbb{Q}(x_i^{(j)})$, let s_i act on A entrywise.

Proposition 2.12. *We have*

$$M(\mathbf{x}_i)M(\mathbf{x}_{i+1}) = s_i(M(\mathbf{x}_i)M(\mathbf{x}_{i+1})) \tag{1}$$

for each $i \leq m-1$. Thus, loop elementary (and homogenous) symmetric functions in the variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are invariant under the action of each s_i.

Proof. For ease of notation in proving (1), replace $\mathbf{x}_i, \mathbf{x}_{i+1}, s_i$ with $\mathbf{x}, \mathbf{y}, s$. Let $A = M(\mathbf{x}_i)M(\mathbf{x}_{i+1})$. Since whirls are upper triangular with ones on the main diagonal, the same is true of A. The only other nonzero entries of A are $A_{i,i+1} = x^{(i)} + y^{(i)}$ and $A_{i,i+2} = x^{(i)}y^{(i+1)}$. So to show that $s(A) = A$, it suffices to show that $x^{(i)} + y^{(i)} = s(x^{(i)} + y^{(i)})$ and $x^{(i)}y^{(i+1)} = s(x^{(i)}y^{(i+1)})$. The second equation is easy:

$$s(x^{(i)}y^{(i+1)}) = y^{(i+1)}\frac{\kappa^{(i+1)}}{\kappa^{(i)}}x^{(i)}\frac{\kappa^{(i)}}{\kappa^{(i+1)}} = y^{(i+1)}x^{(i)}.$$

For the first equation, we have

$$\begin{aligned}
s(x^{(i)} + y^{(i)}) \cdot \kappa^{(i)} &= y^{(i+1)}\kappa^{(i+1)} + x^{(i-1)}\kappa^{(i-1)} \\
&= y^{(i+1)}(x^{(i+2)}x^{(i+3)} \cdots x^{(i)} + y^{(i+2)}x^{(i+3)} \cdots x^{(i)} + \cdots + y^{(i+2)}y^{(i+3)} \cdots y^{(i)}) \\
&\quad + x^{(i-1)}(x^{(i)}x^{(i+1)} \cdots x^{(i-2)} + y^{(i)}x^{(i+1)} \cdots x^{(i-2)} + \cdots + y^{(i)}y^{(i+1)} \cdots y^{(i-2)}) \\
&= (x^{(i)} + y^{(i)})(x^{(i+1)}x^{(i+2)} \cdots x^{(i-1)} + y^{(i+1)}x^{(i+2)} \cdots x^{(i-1)} + \cdots + y^{(i+1)}y^{(i+2)} \cdots y^{(i-1)}) \\
&= (x^{(i)} + y^{(i)}) \cdot \kappa^{(i)}
\end{aligned}$$

and dividing by $\kappa^{(i)}$ gives the desired equality.

Since loop elementary symmetric functions are defined to be the entries of $M(\mathbf{x}_1) \cdots M(\mathbf{x}_m)$, it follows from (1) that they are invariant under the action of each s_i. For the claim about loop homogeneous symmetric functions, note that

$$N(\mathbf{x}_{i+1})N(\mathbf{x}_i) = (M(\mathbf{x}_i)M(\mathbf{x}_{i+1}))^{-c} = s_i(M(\mathbf{x}_i)M(\mathbf{x}_{i+1}))^{-c} = s_i(N(\mathbf{x}_{i+1})N(\mathbf{x}_i)). \tag{2}$$

The equality of the first and last terms in (2) shows that the entries of $N(\mathbf{x}_m) \cdots N(\mathbf{x}_1)$ are invariant under the action of each s_i, and these entries are precisely the loop homogeneous symmetric functions. ∎

We would like to be able to apply a sequence of maps s_i to the variables $x_i^{(j)}$. The following proposition states the crucial properties of the s_i in this regard.

Proposition 2.13. *The maps s_i satisfy*
(a) $s_i^2 = 1$.
(b) If $|i-j| \geq 2$, *then* $s_is_j = s_js_i$.
(c) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$.

Proof. Since $\kappa_i^{(r)} = h_{n-1}^{(r+1)}(\mathbf{x}_i, \mathbf{x}_{i+1})$ is a loop homogeneous symmetric function in the variables $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, we have

$$s_i(\kappa_i^{(r)}) = \kappa_i^{(r)} \tag{3}$$

by Proposition 2.12. Thus,

$$s_i^2(x_i^{(j)}) = s_i(x_{i+1}^{(j+1)}) \cdot s_i\left(\frac{\kappa_i^{(j+1)}}{\kappa_i^{(j)}}\right) = x_i^{(j)}\frac{\kappa_i^{(j)}}{\kappa_i^{(j+1)}} \cdot \frac{\kappa_i^{(j+1)}}{\kappa_i^{(j)}} = x_i^{(j)}.$$

A similar computation shows that s_i^2 fixes $x_{i+1}^{(j)}$, and obviously s_i^2 fixes $x_k^{(j)}$ for $k \neq i, i+1$. This proves (a). (Note: a different proof is given in [LP12].)

Part (b) is immediate from the definition.

We omit the proof of part (c). A number of independent proofs of this "Yang-Baxter relation" exist in the literature: see for example [LP12], [LP13a], [Yam01], [Eti03], [BKa] and [BKb]. □

This proposition shows that the maps s_i satisfy the relations of the simple generators of the symmetric group S_m, and thus they generate an action of S_m on $\mathbb{Q}(x_i^{(j)})$. We will call this action the *birational S_m action.*

Remark 2.14. There is an ambiguity in the definition of this action. Here we have defined $s_1 s_2 = s_1 \circ s_2$, whereas in [LP12] and [LP13a], $s_1 s_2$ is defined by first changing $\mathbf{x_2}$ and $\mathbf{x}_3$ into $\mathbf{x}_2'$ and $\mathbf{x}_3'$ by the action of s_2, and then changing $\mathbf{x}_1$ and $\mathbf{x}_2'$ into $\mathbf{x}_1'$ and $\mathbf{x}_2''$ by the action of s_1. Fortunately, these actions are easily seen to be "inverse" to each other: our $s_1 s_2$ is the same as their $s_2 s_1$. (This is analogous to the inverse relationship between the actions of permutations on an ordered set of m elements by "position" and by "value.") In fact, in this paper we will only consider the action of transpositions $t_{a,b}$, so it does not matter which definition of the action we use.

Corollary 2.15. *The loop elementary (resp., homogeneous) symmetric functions are invariant under the birational S_m action.*

In fact, the loop elementary symmetric functions generate the subring of polynomial invariants for this action ([LP]).

Although the birational S_m action is rather complicated, there is a subring of polynomials for which it reduces to the usual permutation action.

Definition 2.16. For $i = 1, ..., m$, define $\pi_i = x_i^{(1)} x_i^{(2)} \cdots x_i^{(n)}$.

Lemma 2.17. *For $\sigma \in S_m$, we have $\sigma(\pi_i) = \pi_{\sigma(i)}$.*

Proof. Since S_m is generated by the s_i, it suffices to show that $s_i(\pi_j) = \pi_{s_i(j)}$ for each i, j. We have

$$\begin{aligned} s_i(\pi_i) &= s_i(x_i^{(1)}) s_i(x_i^{(2)}) \cdots s_i(x_i^{(n)}) \\ &= x_{i+1}^{(2)} \frac{\kappa_i^{(2)}}{\kappa_i^{(1)}} x_{i+1}^{(3)} \frac{\kappa_i^{(3)}}{\kappa_i^{(2)}} \cdots x_{i+1}^{(1)} \frac{\kappa_i^{(1)}}{\kappa_i^{(n)}} \\ &= \pi_{i+1}. \end{aligned}$$

Similarly, $s_i(\pi_{i+1}) = \pi_i$. And clearly $s_i(\pi_j) = \pi_j$ if $j \neq i, i+1$. □

Definition 2.18. For each positive integer k, define the *loop power sum symmetric function*

$$p_k(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{i=1}^m \pi_i^k = \sum_{i=1}^m \left(\prod_{j=1}^n x_i^{(j)}\right)^k.$$

Note that p_k is homogeneous of degree kn. By Lemma 2.17, the polynomials p_k are invariant under the birational S_m action. In fact, the p_k generate the subring of polynomials which remain polynomial under the birational S_m action ([LP]).

2.4. **Loop Schur Functions.** We will now review the definition of loop Schur functions, which were defined in [LP12]. Note that our conventions follow those of [Lam12], which are the "mirror image" of the conventions in [LP12].

Definition 2.19. Fix a color $r \in \mathbb{Z}/n\mathbb{Z}$. Let λ be a partition with at most m parts, and let T be a tableau of shape λ with entries in $\{1, \ldots, m\}$. If the (i,j)-th box of T contains the number k, then we assign to this box the "colored weight" $x_k^{(r+i-j)}$ (we say that $r+i-j$ is the "color" of the box). We define the colored weight of T with respect to the fixed color r (denoted $\mathrm{cwt}_r(T)$) to be the product of the colored weights of its boxes. We define the *loop Schur function* $s_\lambda^{(r)}$ by

$$s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum \mathrm{cwt}_r(T)$$

where the sum is over all semistandard tableaux T of shape λ with entries in $\{1, \ldots, m\}$.

Example 2.20. Let $m = 2$, $n = 3$, and $\lambda = (3, 2)$. There are two semistandard tableaux of shape λ with entries in $\{1, 2\}$:

If we set $r = 1$ and compute the colored weights of these two tableaux, we find

$$s_{(3,2)}^{(1)} = x_1^{(1)} x_1^{(2)} x_1^{(3)} x_2^{(1)} x_2^{(2)} + x_1^{(1)} x_1^{(3)} x_2^{(1)} (x_2^{(2)})^2.$$

In the tableaux above, red represents the color 1, green represents the color 2, and yellow represents the color 3 (recall that colors are always taken modulo n).

Comparing Definition 2.19 with Proposition 2.7, we obtain the identities

$$s_{(k)}^{(r)} = h_k^{(r-k+1)}$$

and

$$s_{(1^k)}^{(r)} = e_k^{(r)}.$$

The first of these formulas is reminiscent of the Jacobi-Trudi formula for ordinary Schur polynomials, and in fact it is a special case of a Jacobi-Trudi formula for loop Schur functions, which was proven in [LP12].

Proposition 2.21. *For any partition λ with at most m rows, we have*

$$s_\lambda^{(r)} = \det(h_{\lambda_i - i + j}^{(r-\lambda_i+i)}),$$

where all loop symmetric functions are in the variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

Proof. We use the standard Gessel-Viennot lattice path method to evaluate the determinant. Specifically, we consider lattice paths in $\mathbb{Z}^2$ whose edges are directed to the north or east. Each vertical edge has weight one, and a horizontal edge from $(i-1, j)$ to (i, j) has weight $x_j^{(r-i)}$. For a permutation $\sigma \in S_m$, we consider collections of paths starting at $(-\sigma(i), 1)$ and ending at $(\lambda_i - i, m)$. Now the usual involution cancels terms in the determinant coming from non-disjoint collections of paths, and we are left with disjoint collections of paths corresponding to

the identity permutation. Such collections of paths are in bijective correspondence with semi-standard tableaux of shape λ with entries in $\{1, \ldots, m\}$, and it's easy to see that our choice of edge weights will yield the colored tableaux weights that are used to define $s_\lambda^{(r)}$. $\square$

Remark 2.22. The standard proof has been adapted to the loop situation by coloring the columns $\ldots r, r-1, r-2 \ldots$ with the column between -1 and 0 colored r. The colors are decreasing along paths because the loop homogeneous symmetric functions, as defined here, are sums over decreasing sequences of indices. The reason that $r - \lambda_i + i$ appears in the formula instead of $r - j + 1$ is that the loop homogeneous symmetric functions are indexed by their smallest color. The same proof shows that the formula is valid for loop Schur functions in infinitely many variables (just take $m \to \infty$). Also, this proof can easily be extended to the case of skew-shapes.

3. Main Results

3.1. **Ratio of Alternants Formula.** For a sequence $\alpha = (\alpha_1, ..., \alpha_m)$ of nonnegative integers, define the $m \times m$ matrix $A_\alpha^{(r)}$ by

$$(A_\alpha^{(r)})_{ij} = t_{j,m}(h_{\alpha_i}^{(r-\alpha_i+1)}(\mathbf{x}_m)) = t_{j,m}(x_m^{(r)} x_m^{(r-1)} \cdots x_m^{(r-\alpha_i+1)})$$

where $t_{a,b}$ is the transposition in S_m which swaps a and b, acting on $\mathbb{Q}(x_i^{(j)})$ via the birational action. Set $a_\alpha^{(r)} = \det(A_\alpha^{(r)})$; this determinant is called an *alternant.*

Example 3.1. If $m = 2$ and $n = 3$, then

$$A_{(4,2)}^{(2)} = \begin{pmatrix} x_1^{(1)} x_1^{(3)} x_1^{(2)} x_1^{(1)} \frac{x_1^{(2)} x_1^{(3)} + x_2^{(2)} x_1^{(3)} + x_2^{(2)} x_2^{(3)}}{x_1^{(3)} x_1^{(1)} + x_2^{(3)} x_1^{(1)} + x_2^{(3)} x_2^{(1)}} & x_2^{(2)} x_2^{(1)} x_2^{(3)} x_2^{(2)} \\ \\ x_1^{(1)} x_1^{(3)} \frac{x_1^{(1)} x_1^{(2)} + x_2^{(1)} x_1^{(2)} + x_2^{(1)} x_2^{(2)}}{x_1^{(3)} x_1^{(1)} + x_2^{(3)} x_1^{(1)} + x_2^{(3)} x_2^{(1)}} & x_2^{(2)} x_2^{(1)} \end{pmatrix}$$

Theorem 3.2. *Suppose λ is a partition with at most m parts. Then we have*

$$s_\lambda^{(r-m+1)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \frac{a_{\lambda+\delta}^{(r)}}{a_\delta^{(r)}}$$

where $\delta = (m-1, m-2, \ldots, 1, 0)$.

Proof. This proof is modeled on the proof given in [Mac79] of the ratio of alternants formula for ordinary Schur polynomials. Unless otherwise noted, all loop symmetric functions are in the variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

Set $\alpha = \lambda + \delta$. Define an $m \times m$ matrix $H_\alpha^{(r-m+1)}$ by

$$(H_\alpha^{(r-m+1)})_{ij} = h_{\alpha_i - m + j}^{(r-\alpha_i+1)} = h_{\lambda_i - i + j}^{(r-m+1-\lambda_i+i)}.$$

Note that $\det(H_\alpha^{(r-m+1)}) = s_\lambda^{(r-m+1)}$ by Proposition 2.21 above. Define an $m \times m$ matrix $M^{(r)}$ by

$$(M^{(r)})_{ij} = (-1)^{m-i} t_{j,m}(e_{m-i}^{(r+i+1-m)}[m])$$

where the $[m]$ indicates that $\mathbf{x}_m$ is omitted (i.e., $e_k^{(r)}[m] = e_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_{m-1})$). I claim that

$$H_\alpha^{(r-m+1)} M^{(r)} = A_\alpha^{(r)}. \tag{4}$$

This amounts to showing that for each i, j, we have

$$\sum_{k=1}^{m} h_{\alpha_i-m+k}^{(r-\alpha_i+1)}(-1)^{m-k} t_{j,m}(e_{m-k}^{(r+k+1-m)}[m]) = t_{j,m}(h_{\alpha_i}^{(r-\alpha_i+1)}(\mathbf{x}_m)). \tag{5}$$

To this end, set

$$A = N(\mathbf{x}_m) \cdots N(\mathbf{x}_1)$$

and

$$B = M(\mathbf{x}_1)^c \cdots M(\mathbf{x}_{m-1})^c.$$

Clearly we have $AB = N(\mathbf{x}_m)$, so

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = h_{j-i}^{(i)}(\mathbf{x}_m). \tag{6}$$

Let $t_{a,b}$ act entrywise on matrices (via the birational action). If we apply $t_{a,b}$ to equation (6) and use the fact that $t_{a,b}(A) = A$, we get the equation

$$\sum_k A_{ik} t_{a,b}(B_{kj}) = t_{a,b}(h_{j-i}^{(i)}(\mathbf{x}_m)). \tag{7}$$

Using the definition of the $e_k^{(r)}$ and $h_k^{(r)}$, this equation becomes

$$\sum_k h_{k-i}^{(i)}(-1)^{j-k} t_{a,b}(e_{j-k}^{(k)}[m]) = t_{a,b}(h_{j-i}^{(i)}(\mathbf{x}_m)). \tag{8}$$

I will choose indices so that equation (8) turns into equation (5). Take $a = j$ and $b = m$, and set

$$i = r - \alpha_i + 1 \quad \text{and} \quad j = r + 1.$$

Equation (8) becomes

$$\sum_k h_{k-r+\alpha_i-1}^{(r-\alpha_i+1)}(-1)^{r+1-k} t_{j,m}(e_{r+1-k}^{(k)}[m]) = t_{j,m}(h_{\alpha_i}^{(r-\alpha_i+1)}(\mathbf{x}_m)). \tag{9}$$

Now replace k by $k - m + r + 1$, so that equation (9) becomes

$$\sum_k h_{\alpha_i-m+k}^{(r-\alpha_i+1)}(-1)^{m-k} t_{j,m}(e_{m-k}^{(r+k+1-m)}[m]) = t_{j,m}(h_{\alpha_i}^{(r-\alpha_i+1)}(\mathbf{x}_m)). \tag{10}$$

Since $e_{m-k}^{(s)}[m] = 0$ unless $0 \le m - k \le m - 1$, we only need to sum over values of k between 1 and m, so we obtain equation (5). This proves the matrix equation (4).

To complete the proof, observe that $H_\delta^{(r-m+1)}$ is upper triangular with 1's on the diagonal, so it has determinant 1, and thus equation (4) with $\alpha = \delta$ implies

$$\det(M^{(r)}) = \det(A_\delta^{(r)}) = a_\delta^{(r)}.$$

Taking determinants of equation (4) and recalling that $\det(H_{\lambda+\delta}^{(r-m+1)}) = s_\lambda^{(r-m+1)}$, we obtain

$$s_\lambda^{(r-m+1)} a_\delta^{(r)} = a_{\lambda+\delta}^{(r)}.$$

$\square$

Example 3.3. When $m = 1$, the alternants formula becomes

$$s_{(k)}^{(r)}(\mathbf{x}_1) = h_k^{(r-k+1)}(\mathbf{x}_1),$$

an identity which was stated right after Example 2.20.

For a nontrivial example, set $m = 2$, $n = 3$, and $\lambda = (3, 2)$. We have

$$\begin{aligned}\frac{a_{\lambda+\delta}^{(2)}}{a_\delta^{(2)}} &= \frac{x_1^{(1)}x_1^{(3)}x_1^{(2)}x_1^{(1)}(x_1^{(2)}x_1^{(3)} + x_2^{(2)}x_1^{(3)} + x_2^{(2)}x_2^{(3)})x_2^{(2)}x_2^{(1)} - x_1^{(1)}x_1^{(3)}(x_1^{(1)}x_1^{(2)} + x_2^{(1)}x_1^{(2)} + x_2^{(1)}x_2^{(2)})x_2^{(2)}x_2^{(1)}x_2^{(3)}x_2^{(2)}}{x_1^{(1)}(x_1^{(2)}x_1^{(3)} + x_2^{(2)}x_1^{(3)} + x_2^{(2)}x_2^{(3)}) - x_2^{(2)}(x_1^{(3)}x_1^{(1)} + x_2^{(3)}x_1^{(1)} + x_2^{(3)}x_2^{(1)})} \\ &= \frac{(\pi_1)^2x_2^{(1)}x_2^{(2)} + \pi_1 x_1^{(1)}x_1^{(3)}x_2^{(1)}(x_2^{(2)})^2 - \pi_1\pi_2 x_2^{(1)}x_2^{(2)} - x_1^{(1)}x_1^{(3)}\pi_2 x_2^{(1)}(x_2^{(2)})^2}{\pi_1 - \pi_2} \\ &= x_1^{(1)}x_1^{(2)}x_1^{(3)}x_2^{(1)}x_2^{(2)} + x_1^{(1)}x_1^{(3)}x_2^{(1)}(x_2^{(2)})^2 \\ &= s_{(3,2)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2)\end{aligned}$$

(c.f. Example 2.20).

3.2. **Loop Murnaghan-Nakayama Rule.** In this section, we assume that the reader is familiar with the ordinary Murnaghan-Nakayama rule and its proof using the ordinary ratio of alternants formula (see, for example, [Sta99, 7.17]). We show here that the "loop Murnaghan-Nakayama rule" follows from the loop ratio of alternants formula (Theorem 3.2) in a similar manner. Recall the definition of the loop power sum symmetric functions (c.f. 2.18).

Theorem 3.4. *Let $\ell(\lambda)$ be the number of nonzero parts in λ, and let k be a positive integer. For $m \geq \ell(\lambda) + kn$, we have*

$$p_k(\mathbf{x}_1, \ldots, \mathbf{x}_m)s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$$

where the sum is over all partitions μ obtained from λ by adding a border strip of size kn, and $\mathrm{ht}(\mu/\lambda)$ *is one less than the number of rows in the border strip μ/λ.*

Proof. Due to Theorem 3.2, it suffices to show that $p_k a_{\lambda+\delta}^{(r+m-1)} = \sum (-1)^{\mathrm{ht}(\mu/\lambda)} a_{\mu+\delta}^{(r+m-1)}$ whenever $m \geq \ell(\lambda) + nk$. By definition, we have

$$p_k a_{\lambda+\delta}^{(r+m-1)} = \left(\sum_{i=1}^m \pi_i^k\right)\left(\sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\prod_{j=1}^m t_{\sigma(j),m}(h_{\lambda_j+m-j}^{(r-\lambda_j+j)}(\mathbf{x}_m))\right)$$

$$= \sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\sum_{i=1}^m \pi_{\sigma(i)}^k \prod_{j=1}^m t_{\sigma(j),m}(h_{\lambda_j+m-j}^{(r-\lambda_j+j)}(\mathbf{x}_m)). \tag{11}$$

Since $t_{a,b}(\pi_a^k) = \pi_b^k$, we can rewrite (11) as

$$\sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\sum_{i=1}^m t_{\sigma(i),m}(\pi_m^k \cdot h_{\lambda_i+m-i}^{(r-\lambda_i+i)}(\mathbf{x}_m))\prod_{j\neq i} t_{\sigma(j),m}(h_{\lambda_j+m-j}^{(r-\lambda_j+j)}(\mathbf{x}_m)). \tag{12}$$

Now observe that since $\pi_m^k = (x_m^{(1)})^k \cdots (x_m^{(n)})^k$, we have

$$\pi_m^k \cdot h_a^{(b)}(\mathbf{x}_m) = h_{a+nk}^{(b)}(\mathbf{x}_m).$$

This allows us to rewrite (12) as

$$\sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\sum_{i=1}^m\prod_{j=1}^m t_{\sigma(j),m}(h_{\lambda_j+m-j+kn\cdot\delta_{i,j}}^{(r-\lambda_j+j)}(\mathbf{x}_m))$$

where $\delta_{i,j}$ is the Kronecker delta symbol. Switching the order of summation once again, we obtain

$$\sum_{i=1}^{m} a^{(r+m-1)}_{\lambda+\delta+kn\epsilon_i} \tag{13}$$

where ϵ_i is the ith standard basis vector in $\mathbb{Z}^m$.

At this point, the proof of the Murnaghan-Nakayama rule for ordinary symmetric polynomials can be applied with no changes, other than the inclusion of a superscript on each alternant. Let μ_i be the partition obtained from $\lambda + \delta + kn\epsilon_i$ by rearranging the entries in decreasing order. If μ_i has two equal parts, then clearly $a^{(r+m-1)}_{\mu_i} = 0$; if μ_i is strictly decreasing, then $a^{(r+m-1)}_{\mu_i} = (-1)^l a^{(r+m-1)}_{\lambda+\delta+kn\epsilon_i}$, where l is the number of swaps that must be made to rearrange $\lambda + \delta + kn\epsilon_i$ into decreasing order.

Observe that for each $i \leq m$, there is at most one border strip ρ_i of size kn that can be added to λ, whose bottom-left box is in the ith row. The rows that admit such border strips are precisely the rows i for which μ_i contains no equal parts. Furthermore, the partition $\lambda+\rho_i+\delta$ is equal to μ_i, which means in particular that the height of ρ_i is equal to the number of swaps which must be made to turn $\lambda+\delta+kn\epsilon_i$ into μ_i. (Here we use the hypothesis that $m \geq \ell(\lambda)+kn$: if $m < \ell(\lambda) + kn$, then the sum in (13) would miss the border strip starting in row $\ell(\lambda) + kn$, which consists of a column of kn boxes beneath the diagram of λ.) This argument shows that

$$\sum_{i=1}^{m} a^{(r+m-1)}_{\lambda+\delta+kn\epsilon_i} = \sum (-1)^{\mathrm{ht}(\mu/\lambda)} a^{(r+m-1)}_{\mu}$$

where the second sum is over all partitions μ such that μ/λ is a border strip of size kn. And this is precisely what we needed to show.

□

Remark 3.5. The key to this proof is the fact that when we multiply $h^{(b)}_a(\mathbf{x}_m)$ by π^k_m, the color index "b" does not change (which is due to the fact that each color appears the same number of times in π^k_m). We cannot use a similar argument to prove a "loop Pieri rule" because the color indices do not behave nicely when we multiply an alternant by an arbitrary monomial which appears in a loop elementary or homogeneous symmetric function. Indeed, we do not know of any loop analogue of the Pieri rule.

References

[BKa] A. Berenstein and D. Kazhdan. Geometric and unipotent crystals. *Geom. Funct. Anal.*, 2000, Special Volume, Part I:188–236.

[BKb] A. Braverman and D. Kazhdan. γ-functions of representations and lifting (with an appendix by V. Vologodsky). *Geom. Func. Anal.*, 2000, Special Volume, Part I:237–278.

[Eti03] P. Etingof. Geometric crystals and set-theoretical solutions to the quantum Yang-Baxter equation. *Comm. Algebra*, 31(4):1961–1973, 2003.

[HHI$^+$01] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi, and T. Tokihiro. The $A_M^{(1)}$ automata related to crystals of symmetric tensors. *J. Math. Phys.*, 42(1):274–308, 2001.

[Lam12] T. Lam. Loop symmetric functions and factorizing matrix polynomials. *Fifth international congress of Chinese mathematicians, AMS/IP studies in advanced mathematics*, 51:609–628, 2012.

[LP] T. Lam and P. Pylyavskyy. Private communication.

[LP12] T. Lam and P. Pylyavskyy. Total positivity for loop groups I: whirls and curls. *Adv. Math.*, 230(3):1222–1271, 2012.

[LP13a] T. Lam and P. Pylyavskyy. Crystals and total positivity on orientable surfaces. *Selecta Math.*, 19(1):173–235, 2013.

[LP13b] T. Lam and P. Pylyavskyy. Intrinsic energy is a loop Schur function. *J. Comb. 4*, 4:387–401, 2013.

[LPS] T. Lam, P. Pylyavskyy, and R. Sakamoto. Rigged configurations and cylindric loop Schur functions, preprint (arxiv:1410.4455v1).

[Mac79] I.G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 1979.

[Ros14] D. Ross. The Loop Murnaghan-Nakayama Rule. *J. Algebraic Combin.*, 39(1):3–15, 2014.

[Sta99] R. P. Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge University Press, 1999.

[Yam01] Y. Yamada. *A birational representation of Weyl group, combinatorial R-matrix and discrete Toda equation*. Physics and combinatorics, 2000 (Nagoya), 305-319, World Sci. Publ., River Edge, NJ, 2001.

E-mail address: gfrieden@umich.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 USA